Formulas and equations for finding scattering data from the Dirichlet-to-Neumann map with nonzero background potential

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Abstract

For the Schrödinger equation at fixed energy with a potential supported in a bounded domain we give formulas and equations for finding scattering data from the Dirichlet-to-Neumann map with nonzero background potential. For the case of zero background potential these results were obtained in [R.G.Novikov, Multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$, Funkt. Anal. i Ego Prilozhen 22(4), pp.11-22, (1988)].

1.Introduction

Consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D, \tag{1.1}$$

where

$$\begin{aligned} &D \ \text{ is an open bounded domain in } \ \mathbb{R}^d, \quad d \geq 2,\\ &\text{with } \ \partial D \in C^2, \end{aligned} \tag{1.2a}$$

$$v \in L^\infty(D). \tag{1.2b}$$

We also assume that

$$\begin{aligned} &E \ \text{ is not a Dirichlet eigenvalue for}\\ &\text{the operator } \ -\Delta + v \ \text{ in } \ D. \end{aligned} \tag{1.3}$$

Consider the map $\Phi(E)$ such that

$$\frac{\partial\psi}{\partial\nu}\big|_{\partial D} = \Phi(E)\big(\psi\big|_{\partial D}\big) \tag{1.4}$$

for all sufficiently regular solutions of (1.1) in $\bar D = D \cup \partial D$, for example, for all $\psi \in H^1(D)$ satisfying (1.1), where ν is the outward normal to ∂D. The map $\Phi(E)$ is called the Dirichlet-to-Neumann map for equation (1.1).

Consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \tag{1.5}$$

where

$$\rho^{d+\varepsilon} v \in L^\infty(\mathbb{R}^d), \quad d \geq 2, \quad \text{for some } \ \varepsilon > 0, \tag{1.6}$$

where ρ denotes the multiplication operator by the function $\rho(x) = 1 + |x|$. For equation (1.5) we consider the functions ψ^+ and f of the classical scattering theory and the Faddeev functions ψ, h, ψ_γ, h_γ (see, for example, [F1], [F2], [F3], [HN], [Ne]).

The functions ψ^+ and f are defined as follows:

$$\psi^+(x,k) = e^{ikx} + \int_{\mathbb{R}^d} G^+(x-y,k)v(y)\psi^+(y,k)dy, \tag{1.7}$$

$$G^+(x,k) = -\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \frac{e^{i\xi x}d\xi}{\xi^2 - k^2 - i0}, \tag{1.8}$$

where $x, k \in \mathbb{R}^d$, $k^2 > 0$ (and at fixed k the formula (1.7) is an equation for ψ^+ in $L^\infty(\mathbb{R}^d)$);

$$f(k,l) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{-ilx}v(x)\psi^+(x,k)dx, \tag{1.9}$$

where $k, l \in \mathbb{R}^d$, $k^2 > 0$. Here $\psi^+(x,k)$ satisfies (1.5) for $E = k^2$ and describes scattering of the plane waves e^{ikx}; $f(k,l)$, $k^2 = l^2$, is the scattering amplitude for equation (1.5) for $E = k^2$. The equation (1.7) is called the Lippman-Schwinger integral equation.

The functions ψ and h are defined as follows:

$$\psi(x,k) = e^{ikx} + \int_{\mathbb{R}^d} G(x-y,k)v(y)\psi(y,k)dy, \tag{1.10}$$

$$G(x,k) = e^{ikx}g(x,k), \quad g(x,k) = -\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \frac{e^{i\xi x}d\xi}{\xi^2 + 2k\xi}, \tag{1.11}$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{C}^d \backslash \mathbb{R}^d$ (and at fixed k the formula (1.10) is an equation for $\psi = e^{ikx}\mu(x,k)$, where μ is sought in $L^\infty(\mathbb{R}^d)$);

$$h(k,l) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{-ilx}v(x)\psi(x,k)dx, \tag{1.12}$$

where $k, l \in \mathbb{C}^d \backslash \mathbb{R}^d$, $Im\, k = Im\, l$. Here $\psi(x,k)$ satisfies (1.5) for $E = k^2$, and ψ, G and h are (nonanalytic) continuations of ψ^+, G^+ and f to the complex domain. In particular, $h(k,l)$ for $k^2 = l^2$ can be considered as the "scattering" amplitude in the complex domain for equation (1.5) for $E = k^2$. The functions ψ_γ and h_γ are defined as follows:

$$\psi_\gamma(x,k) = \psi(x, k + i0\gamma), \quad h_\gamma(k,l) = h(k + i0\gamma, l + i0\gamma), \tag{1.13}$$

where $x, k, l, \gamma \in \mathbb{R}^d$, $|\gamma| = 1$. Note that

$$\psi^+(x,k) = \psi_{k/|k|}(x,k), \quad f(k,l) = h_{k/|k|}(k,l), \tag{1.14}$$

where $x, k, l \in \mathbb{R}^d$, $|k| > 0$.

We consider $f(k,l)$ and $h_\gamma(k,l)$, where $k, l, \gamma \in \mathbb{R}^d$, $k^2 = l^2 = E$, $\gamma^2 = 1$, and $h(k,l)$, where $k, l \in \mathbb{C}^d \backslash \mathbb{R}^d$, $Im\, k = Im\, l$, $k^2 = l^2 = E$, as scattering data S_E for equation (1.5) at

fixed $E \in]0,+\infty[$. We consider $h(k,l)$, where $k,l \in \mathbb{C}^d \backslash \mathbb{R}^d$, $Im\,k = Im\,l$, $k^2 = l^2 = E$, as scattering data S_E for equation (1.5) at fixed $E \in \mathbb{C} \backslash]0,+\infty[$.

Let D be a fixed domain satisfying (1.2a). Let

$$v \in L^\infty(D) \ \text{ and } \ v \equiv 0 \ \text{ on } \ \mathbb{R}^d \backslash \bar{D}. \tag{1.15}$$

For v of (1.15) we consider the Dirichlet-to-Neumann map $\Phi(E)$ for equation (1.1) and the scattering data S_E for equation (1.5).

In the present work we continue studies of [No1] on the following inverse boundary value problem for equation (1.1):

Problem 1. Find v (in (1.1)) from $\Phi(E)$ (where E is fixed or belongs to some set).

More precisely, we develop formulas and equations of [No1] which reduce Problem 1 to the following inverse scattering problem for equation (1.5):

Problem 2. Find v (in (1.5)) from $S(E)$ (where E is fixed or belongs to some set).

Concerning results given in the literature on Problem 1, see [SU], [No1], [A], [NSU], [Na1], [Na2], [M] and references therein. Concerning results given in the literature on Problem 2, see [BC], [HN], [No2], [No3], [IS], [GN], [No4], [No5], [E], [Ch], [BBMRS], [BMR] and references therein.

The main results of the present work consist of Theorem 1 and Propositions 1 and 2 of Section 2. In these results we consider for fixed E two potentials v_0 and v satisfying (1.15) and (1.3). In Theorem 1 and Proposition 1 we give formulas and equations for finding S_E from $\Phi(E) - \Phi_0(E)$ and from (some functions found from) v_0, where S_E and $\Phi(E)$ correspond to v and $\Phi_0(E)$ corresponds to v_0. In Proposition 2 we give a result on the solvability of equations of Theorem 1.

For the case when $v_0 \equiv 0$, Theorem 1 and Propositions 1 and 2 were obtained for the first time in [No1] (see also [HN], [Na1], [Na2]), where using these results Problem 1 was reduced to Problem 2. For the case when the best known approximation v_0 to v of Problem 1 is not identically zero (and especially when v_0 is really close to v), results of the present work reduce Problem 1 to Problem 2 in a more stable way than it was done in [No1].

Note that generalizing results of [No1] to the case of nonzero background potential v_0 we used essentially results of [No4]. The main results of the present work are presented in detail in Section 2.

Note, finally, that by the present work we start a development of new exact reconstruction algorithms for Problem 1.

2. Main results

To formuate our results we need to introduce some additional notations. Consider (under the assumption (1.6)) the sets $\mathcal{E}$, $\mathcal{E}_\gamma$, $\mathcal{E}^+$ defined as follows:

$$\begin{aligned}\mathcal{E} = \{\zeta \in \mathbb{C}^d \backslash \mathbb{R}^d \ : \ \ &\text{equation (1.10) for } k = \zeta \text{ is not} \\ &\text{uniquely solvable for } \psi = e^{ikx}\mu \text{ with } \mu \in L^\infty(\mathbb{R}^d)\},\end{aligned} \tag{2.1a}$$

$$\begin{aligned}\mathcal{E}_\gamma = \{\zeta \in \mathbb{R}^d \backslash 0 \ : \ \ &\text{equation (1.10) for } k = \zeta + i0\gamma \text{ is not} \\ &\text{uniquely solvable for } \psi \in L^\infty(\mathbb{R}^d)\}, \ \ \gamma \in \mathbb{S}^{d-1},\end{aligned} \tag{2.1b}$$

$$\mathcal{E}^+ = \{\zeta \in \mathbb{R}^d \backslash 0 \ : \ \ \text{equation} \ \ (1.7) \ \ \text{for} \ k = \zeta \ \ \text{is not} \\ \text{uniquely solvable for} \ \ \psi^+ \in L^\infty(\mathbb{R}^d)\}. \tag{2.1c}$$

Note that $\mathcal{E}^+$ is a well-known set of the classical scattering theory for equation (1.5) and that $\mathcal{E}^+ = \emptyset$ for real-valued v satisfying (1.6) (see, for example, [Ne]). The sets $\mathcal{E}$ and $\mathcal{E}_\gamma$ were considered for the first time in [F1], [F2], [F3]. Concerning the properties of $\mathcal{E}$ and $\mathcal{E}_\gamma$, see [F3], [HN], [LN], [Ne], [We], [Na2], [No4], [No6].

Consider (under the assumptions (1.6)) the functions R, R_γ, R^+ defined as follows:

$$R(x,y,k) = G(x-y,k) + \int_{\mathbb{R}^d} G(x-z,k)v(z)R(z,y,k)dz, \tag{2.2}$$

where $x, y \in \mathbb{R}^d$, $k \in \mathbb{C}^d \backslash \mathbb{R}^d$, G is defined by (1.11), and at fixed y and k the formula (2.2) is an equation for

$$R(x,y,k) = e^{ik(x-y)} r(x,y,k), \tag{2.3}$$

where r is sought with the properties

$$r(\cdot, y, k) \ \ \text{is continuous on} \ \ \mathbb{R}^d \backslash y, \tag{2.4a}$$

$$r(x,y,k) \to 0 \ \ \text{as} \ \ |x| \to \infty, \tag{2.4b}$$

$$r(x,y,k) = O(|x-y|^{2-d}) \ \ \text{as} \ \ x \to y \ \ \text{for} \ \ d \geq 3, \\ r(x,y,k) = O(|\ln |x-y||) \ \ \text{as} \ \ x \to y \ \ \text{for} \ \ d = 2; \tag{2.4c}$$

$$R_\gamma(x,y,k) = R(x,y,k + i0\gamma), \tag{2.5}$$

where $x, y \in \mathbb{R}^d$, $k \in \mathbb{R}^d \backslash 0$, $\gamma \in \mathbb{S}^{d-1}$;

$$R^+(x,y,k) = R_{k/|k|}(x,y,k), \tag{2.6}$$

where $x, y \in \mathbb{R}^d$, $k \in \mathbb{R}^d \backslash 0$. Note that $R(x,y,k)$, $R_\gamma(x,y,k)$ and $R^+(x,y,k)$ (for their domains of definition in k and γ) satisfy the equation

$$(\Delta + E - v(x))R(x,y,k) = \delta(x-y), \ \ x \in \mathbb{R}^d, \ \ y \in \mathbb{R}^d, \ \ E = k^2. \tag{2.7}$$

The function $R^+(x,y,k)$ (defined by means of (2.2) for $k \in \mathbb{R}^d \backslash 0$ with G replaced by G^+ of (1.8)) is well-known in the scattering theory for equations (1.5), (2.7). In particular, this function describes scattering of the spherical waves $G^+(x-y,k)$ generated by a source at y. Apparently, the functions R and R_γ were considered for the first time in [No5].

Note that under the assumption (1.6): equation (2.2) at fixed y and k is uniquely solvable for R with the properties (2.3), (2.4) if and only if $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E})$; equation (2.2) with $k = \zeta + i0\gamma$, $\zeta \in \mathbb{R}^d \backslash 0$, $\gamma \in \mathbb{S}^{d-1}$, at fixed y, ζ and γ is uniquely solvable for R_γ if and only if $\zeta \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}_\gamma)$; equation (2.2) with $k = \zeta + i0\zeta/|\zeta|$, $\zeta \in \mathbb{R}^d \backslash 0$, at fixed y and ζ is uniquely solvable for R^+ if and only if $\zeta \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}^+)$.

For v of (1.15) we consider the map $\Phi(E)$ defined by means of (1.4), the functions ψ^+, f, ψ, h, ψ_γ, h_γ and R^+, R, R_γ defined by means of (1.7)-(1.13) and (2.2)-(2.6) and

the sets $\mathcal{E}$, $\mathcal{E}_\gamma$, $\mathcal{E}^+$ defined by (2.1). The Schwartz kernel of the integral operator $\Phi(E)$ will be denoted by $\Phi(x,y,E)$, where $x,y\in\partial D$.

Theorem 1. *Let D satisfying* (1.2a) *and E be fixed. Let v^0 and v be two potentials satisfying* (1.15), (1.3). *Let Φ, ψ^+, f, ψ, h, ψ_γ, h_γ, R^+, R, R_γ, $\mathcal{E}$, $\mathcal{E}_\gamma$, $\mathcal{E}^+$ correspond to v (as defined above) and Φ^0, $\psi^{+,0}$, f^0, ψ^0, h^0, ψ^0_γ, h^0_γ, $R^{+,0}$, R^0, R^0_γ, $\mathcal{E}^0$, $\mathcal{E}^0_\gamma$, $\mathcal{E}^{+,0}$ correspond to v^0 (as defined above with $v=v^0$). Then the following formulas hold:*

$$h(k,l)=h^0(k,l)+\big(\frac{1}{2\pi}\big)^d\int_{\partial D}\int_{\partial D}\psi^0(x,-l)(\Phi-\Phi^0)(x,y,E)\psi(y,k)dydx \tag{2.8}$$

for $k,l\in\mathbb{C}^d\backslash(\mathbb{R}^d\cup\mathcal{E}^0\cup\mathcal{E})$, $Im\,k=Im\,l$, $k^2=l^2=E$,

$$\psi(x,k)=\psi^0(x,k)+\int_{\partial D}A(x,y,k)\psi(y,k)dy,\quad x\in\partial D, \tag{2.9a}$$

$$A(x,y,k)=\int_{\partial D}R^0(x,z,k)(\Phi-\Phi^0)(z,y,E)dz,\quad x,y\in\partial D, \tag{2.9b}$$

for $k\in\mathbb{C}^d\backslash(\mathbb{R}^d\cup\mathcal{E}^0\cup\mathcal{E})$, $k^2=E$;

$$h_\gamma(k,l)=h^0_\gamma(k,l)+\big(\frac{1}{2\pi}\big)^d\int_{\partial D}\int_{\partial D}\psi^0_{-\gamma}(x,-l)(\Phi-\Phi^0)(x,y,E)\psi_\gamma(y,k)dydx \tag{2.10}$$

for $k,l\in\mathbb{R}^d\backslash(0\cup\mathcal{E}^0_\gamma\cup\mathcal{E}_\gamma)$, $\gamma\in\mathbb{S}^{d-1}$, $k^2=l^2=E$, $k\gamma=l\gamma$,

$$\psi_\gamma(x,k)=\psi^0_\gamma(x,k)+\int_{\partial D}A_\gamma(x,y,k)\psi_\gamma(y,k)dy,\quad x\in\partial D, \tag{2.11a}$$

$$A_\gamma(x,y,k)=\int_{\partial D}R^0_\gamma(x,z,k)(\Phi-\Phi^0)(z,y,E)dz,\quad x,y\in\partial D, \tag{2.11b}$$

for $k\in\mathbb{R}^d\backslash(0\cup\mathcal{E}^0_\gamma\cup\mathcal{E}_\gamma)$, $\gamma\in\mathbb{S}^{d-1}$, $k^2=E$;

$$f(k,l)=f^0(k,l)+\big(\frac{1}{2\pi}\big)^d\int_{\partial D}\int_{\partial D}\psi^{+,0}(x,-l)(\Phi-\Phi^0)(x,y,E)\psi^+(y,k)dydx \tag{2.12}$$

for $k,l\in\mathbb{R}^d\backslash(0\cup\mathcal{E}^{+,0}\cup\mathcal{E}^+)$, $k^2=l^2=E$,

$$\psi^+(x,k)=\psi^{+,0}(x,k)+\int_{\partial D}A^+(x,y,k)\psi^+(y,k)dy,\quad x\in\partial D, \tag{2.13a}$$

$$A^+(x,y,k)=\int_{\partial D}R^+(x,z,k)(\Phi-\Phi^0)(z,y,E)dz,\quad x,y\in\partial D, \tag{2.13b}$$

for $k\in\mathbb{R}^d\backslash(0\cup\mathcal{E}^{+,0}\cup\mathcal{E}^+)$, $k^2=E$.

Note that in Theorem 1 dx and dy denote the standard measure on ∂D in $\mathbb{R}^d$.

Note that in the formula (2.10) for $h_\gamma(k,l)$ there is the additional restriction: $k\gamma = l\gamma$. To extend (2.10) to the general case, consider $\psi_\gamma(x,k,l)$ defined as follows:

$$\psi_\gamma(x,k,l) = e^{ilx} + \int_{\mathbb{R}^d} G_\gamma(x-y,k)v(y)\psi_\gamma(y,k,l)dy, \tag{2.14a}$$

$$G_\gamma(x,k) = G(x, k+i0\gamma), \tag{2.14b}$$

where $x,k,l \in \mathbb{R}^d$, $k^2 = l^2 > 0$, $\gamma \in \mathbb{S}^{d-1}$ and (2.14) at fixed γ, k, l is an equation for $\psi_\gamma(\cdot,k,l)$ in $L^\infty(\mathbb{R}^d)$.

Proposition 1. *Let the assumptions of Theorem 1 be valid. In addition, let $\psi_\gamma(x,k,l)$ correspond to v and $\psi^0_\gamma(x,k,l)$ correspond to v^0. Then*

$$h_\gamma(k,l) = h^0_\gamma(k,l) + \left(\frac{1}{2\pi}\right)^d \int_{\partial D}\int_{\partial D} \psi^0_{-\gamma}(x,-k,-l)(\Phi-\Phi^0)(x,y,E)\psi_\gamma(y,k)dydx \tag{2.15}$$

for $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d\backslash(0 \cup \mathcal{E}^0_\gamma \cup \mathcal{E}_\gamma)$, $l \in \mathbb{R}^d$, $k^2 = l^2 = E$.

Note that (see [F3], [No4])

$$G_\gamma(x,k) = G_\gamma(x,l) \ \text{ for } \ x,k,l \in \mathbb{R}^d, \quad \gamma \in \mathbb{S}^{d-1}, \quad k^2 = l^2 > 0, \quad k\gamma = l\gamma. \tag{2.16}$$

Therefore,

$$\psi_\gamma(x,k,l) = \psi_\gamma(x,l,l) = \psi_\gamma(x,l) \ \text{ for } \ x,k,l \in \mathbb{R}^d, \quad \gamma \in \mathbb{S}^{d-1}, \quad k^2 = l^2 > 0, \quad k\gamma = l\gamma. \tag{2.17}$$

Therefore, (2.15), under the additional restriction $k\gamma = l\gamma$ is reduced to (2.10).

Suppose that v is unknown, but v^0 and $\Phi(E) - \Phi^0(E)$ are known (v^0 is considered as the best known approximation to v). Then Theorem 1 and Proposition 1 (and equations and formulas (1.7)-(1.13), (2.2)-(2.6), (2.14) for finding $\psi^{+,0}$, f^0, ψ^0, h^0, ψ^0_γ, h^0_γ, $R^{+,0}$, R^0, R^0_γ from v^0) give a method for finding the scattering data S_E (defined in the introduction) for v from the background potential v^0 and the difference $\Phi(E) - \Phi^0(E)$. In addition, (2.9a), (2.11a), (2.13a) at fixed k and γ are linear integral equations for finding ψ, ψ_γ, ψ^+ on ∂D from ψ^0, ψ^0_γ, $\psi^{+,0}$ on ∂D and A, A_γ, A^+ on $\partial D \times \partial D$ (where A, A_γ, A^+ are given by (2.9b), (2.11b), (2.13b)).

Proposition 2. *Under the assumptions of Theorem 1, equation* (2.9a) *at fixed $k \in \mathbb{C}^d\backslash(\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, equation* (2.11a) *at fixed $\gamma \in \mathbb{S}^{d-1}$ and $k \in \mathbb{R}^d\backslash(0 \cup \mathcal{E}^0_\gamma)$, $k^2 = E$, and equation* (2.13a) *at fixed $k \in \mathbb{R}^d\backslash(0 \cup \mathcal{E}^{+,0})$ are Fredholm linear integral equations of the second kind for ψ, ψ_γ and ψ^+ (respectively) in $L^\infty(\partial D)$ and are uniquely solvable (in this space) if and only if $k \notin \mathcal{E}$ for* (2.9a), *$k \notin \mathcal{E}_\gamma$ for* (2.11a) *and $k \notin \mathcal{E}^+$ for* (2.13a).

Note that

$$\psi(x,k) = e^{ikx}\mu(x,k), \quad \psi^0(x,k) = e^{ikx}\mu^0(x,k), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{C}^d\backslash\mathbb{R}^d, \tag{2.18}$$

where e^{ikx} is an exponentially increasing factor and

$$\mu(x,k) \to 1, \quad \mu^0(x,k) \to 1 \ \text{ as } \ |k| \to \infty \tag{2.19}$$

for $k^2 = E$ at fixed E, where $|k| = \sqrt{(Re\, k)^2 + (Im\, k)^2}$. Therefore, it is convenient to write equation (2.9) of Theorem 1 as follows:

$$\mu(x,k) = \mu^0(x,k) + \int_{\partial D} B(x,y,k)\mu(y,k)dy, \quad x \in \partial D, \tag{2.20a}$$

$$B(x,y,k) = \int_{\partial D} r^0(x,z,k)e^{-ikz}(\Phi - \Phi^0)(z,y,E)dz\, e^{iky}, \quad x,y \in \partial D, \tag{2.20b}$$

for $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0 \cup \mathcal{E})$, where r^0 and R^0 are related by (2.3).

Theorem 1 and Proposition 1 reduce Problem 1 to Problem 2 (these problems are formulated in the introduction).

For the case when $v^0 \equiv 0$, Theorem 1 and Propositions 1 and 2 were obtained in [No1] (see also [Na1], [Na2]). Note that the basic results of [No1], in particular formula (2.8) and equation (2.9) for $v^0 \equiv 0$ and $d = 3$, were presented already in the survey given in [HN].

For the case when the best known approximation v^0 to v is not identically zero (and especially when v^0 is really close to v) Theorem 1 and Proposition 1 give a more convenient (in particular, for the stability analysis) method for reducing Problem 1 to Problem 2 than in [No1]. To explain this more precisely, consider, in particular, the integral operators $B(k)$, $A_\gamma(k)$, $A^+(k)$ (with the Schwartz kernels $B(x,y,k)$, $A_\gamma(x,y,k)$, $A^+(x,y,k)$) of equations (2.20a), (2.11a), (2.13a). To have a simple and stable (with respect to small errors in $\Phi(E)$) method for solving equations (2.20a) (for fixed $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$), (2.11a) (for fixed $\gamma \in \mathbb{S}^{d-1}$ and $k \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}^0_\gamma)$, $k^2 = E$) and (2.13a) (for fixed $k \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}^{+,0})$, $k^2 = E$) it is important to have that

$$\|B(k)\| < \eta, \quad \|A_\gamma(k)\| < \eta, \quad \|A^+(k)\| < \eta, \tag{2.21}$$

respectively, for some $\eta < 1$, where $\|A\|$ is the norm of an operator A (for example) in $L^\infty(\partial D)$. In this case equations (2.20a), (2.11a), (2.13a) are uniquely solvable by the method of successive approximations. In addition, if $\eta \ll 1$, then (2.20a), (2.11a), (2.13a) can be solved in the first approximation as

$$\mu(x,k) \approx \mu^0(x,k), \quad \psi_\gamma(x,k) \approx \psi^0_\gamma(x,k), \quad \psi^+(x,k) \approx \psi^{+,0}(x,k) \tag{2.22}$$

and h, h_γ, f can be determined in the first (nontrivial) approximation as

$$h(k,l) \approx h^0(k,l) + \left(\frac{1}{2\pi}\right)^d \int_{\partial D}\int_{\partial D} \psi^0(x,-l)(\Phi - \Phi^0)(x,y,E)\psi^0(y,k)dydx \tag{2.23a}$$

for $l \in \mathbb{C}^d \backslash \mathbb{R}^d$, $Im\, l = Im\, k$, $l^2 = k^2 = E$ (and for k of $B(k)$ of (2.21)),

$$h_\gamma(k,l) \approx h^0_\gamma(k,l) + \left(\frac{1}{2\pi}\right)^d \int_{\partial D}\int_{\partial D} \psi^0_{-\gamma}(x,-k,-l)(\Phi - \Phi^0)(x,y,E)\psi^0_\gamma(y,k)dydx \tag{2.23b}$$

for $l \in \mathbb{R}^d \backslash \mathcal{E}^0_\gamma$, $l^2 = k^2 = E$ (and for k, γ of $A_\gamma(k)$ of (2.21)),

$$f(k,l) \approx f^0(k,l) + \left(\frac{1}{2\pi}\right)^d \int_{\partial D}\int_{\partial D} \psi^{+,0}(x,-l)(\Phi - \Phi^0)(x,y,E)\psi^{+,0}(y,k)dydx \tag{2.23c}$$

for $l \in \mathbb{R}^d$, $l^2 = k^2 = E$ (and for k of $A^+(k)$ of (2.21)).

Note that the direct problem of finding $\psi^{+,0}$, f^0, ψ^0, h^0, ψ^0_γ, h^0_γ, $R^{+,0}$, R^0, R^0_γ (involved into (2.8)-(2.13), (2.15), (2.20)) from v^0 is (relatively) well understood (in comparison with the problem of solving (2.20a), (2.11a), (2.13a) without the assumptions (2.21)) and is sufficiently stable.

Note that in (2.20a), apparently, unfortunately, almost always

$$\|B(k)\| \to \infty \ \ \text{(exponentially fast) as} \ \ |k| \to \infty \tag{2.24}$$

for $k \in \mathbb{C}^d$, $k^2 = E$ at fixed E, in spite of (2.19).

To have (2.21) for $B(k)$ for maximally large domain in $k \in \mathbb{C}^d \backslash \mathbb{R}^d$, $k^2 = E$, and when $E > 0$ for $A_\gamma(k)$ for maximally large domain in $\gamma \in \mathbb{S}^{d-1}$ and $k \in \mathbb{R}^d$, $k^2 = E$, and for $A^+(k)$ for $k \in \mathbb{R}^d$, $k^2 = E$, it is important to have that $\|\Phi(E) - \Phi^0(E)\|$ is as small as possible. The smallness of $\|\Phi(E) - \Phi^0(E)\|$ follows from the closeness of v^0 to v (for example) in $L^\infty(D)$ for fixed D, v and E, under the conditions (1.2), (1.3).

As soon as Problem 1 is reduced to Problem 2, one can use for solving Problem1 methods of [HN], [No2], [Na2], [IS], [No4], [No5], [E], [BBMRS], [BMR] (and further references given therein).

3. Proofs of Theorem 1 and Propositions 1 and 2

For the case when $v^0 \equiv 0$, Theorem 1 and Propositions 1 and 2 were proved in [No1]. In this section we generalize these proofs of [No1] to the case of nonzero background potential v^0. To this end we use, in particular, some results of [A] and [No4].

Proof of Theorem 1. We proceed from the following formulas and equations (being valid under the assumption (1.6) on v^0 and v):

$$h(k,l) = h^0(k,l) + \Big(\frac{1}{2\pi}\Big)^d \int_{\mathbb{R}^d} \psi^0(x,-l)(v(x) - v^0(x))\psi(x,k)dx \tag{3.1}$$

for $k, l \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0 \cup \mathcal{E})$, $Im\, k = Im\, l$, $k^2 = l^2$,

$$\psi(x,k) = \psi^0(x,k) + \int_{\mathbb{R}^d} R^0(x,y,k)(v(y) - v^0(y))\psi(y,k)dy, \tag{3.2}$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$ (and (3.2) at fixed k is an equation for $\psi = e^{ikx}\mu(x,k)$, where μ is sought in $L^\infty(\mathbb{R}^d)$),

$$h_\gamma(k,l) = h^0_\gamma(k,l) + \Big(\frac{1}{2\pi}\Big)^d \int_{\mathbb{R}^d} \psi^0_{-\gamma}(x,-l)(v(x) - v^0(x))\psi_\gamma(x,k)dx \tag{3.3}$$

for $\gamma \in \mathbb{S}^{d-1}$, $k, l \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}^0_\gamma \cup \mathcal{E}_\gamma)$, $k^2 = l^2$, $k\gamma = l\gamma$.

$$\psi_\gamma(x,k) = \psi^0_\gamma(x,k) + \int_{\mathbb{R}^d} R^0_\gamma(x,y,k)(v(y) - v^0(y))\psi_\gamma(y,k)dy, \tag{3.4}$$

where $x \in \mathbb{R}^d$, $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}^0_\gamma)$ (and (3.4) at fixed γ and k is an equation for ψ_γ in $L^\infty(\mathbb{R}^d)$),

$$f(k,l) = f^0(k,l) + \big(\frac{1}{2\pi}\big)^d \int_{\mathbb{R}^d} \psi^{+,0}(x,-l)(v(x) - v^0(x))\psi^+(x,k)dx \tag{3.5}$$

for $k, l \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}^{+,0} \cup \mathcal{E}^+)$, $k^2 = l^2$,

$$\psi^+(x,k) = \psi^{+,0}(x,k) + \int_{\mathbb{R}^d} R^{+,0}(x,y,k)(v(y) - v^0(y))\psi^+(y,k)dy, \tag{3.6}$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}^{+,0})$ (and (3.6) at fixed k is an equation for ψ^+ in $L^\infty(\mathbb{R}^d)$). (We remind that ψ^+, f, ψ, h, ψ_γ, h_γ were defined in the introduction by means of (1.7)-(1.13).) Equation (3.6) is well-known in the classical scattering theory for the Schrödinger equation (1.5). Formula (3.5) was given, in particular, in [St]. To our knowledge formulas and equations (3.1)-(3.4) were given for the first time in [No4].

Note that, under the assumption (1.6):

$$\begin{aligned}
&(3.2)\ \text{ at fixed }\ k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)\ \text{ is uniquely solvable}\\
&\text{for }\ \psi = e^{ikx}\mu(x,k)\ \text{ with }\ \mu \in L^\infty(\mathbb{R}^d)\ \text{ if and only if }\ k \notin \mathcal{E};\\
&(3.4)\ \text{ at fixed }\ \gamma \in \mathbb{S}^{d-1},\ \ k \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}^0_\gamma)\ \text{ is uniquely solvable}\\
&\text{for }\ \psi_\gamma \in L^\infty(\mathbb{R}^d)\ \text{ if and only if }\ k \notin \mathcal{E}_\gamma;\\
&(3.6)\ \text{ at fixed }\ k \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}^{+,0})\ \text{ is uniquely solvable}\\
&\text{for }\ \psi^+ \in L^\infty(\mathbb{R}^d)\ \text{ if and only if }\ k \notin \mathcal{E}^+.
\end{aligned} \tag{3.7}$$

In a similar way with [No1], (under the assumptions of Theorem 1) formulas and equations (3.1)-(3.6) can be transformed into (2.8)-(2.13) by means of the following Green's formula

$$\begin{aligned}
&\int_D (u_1(x)\Delta u_2(x) - u_2(x)\Delta u_1(x))dx =\\
&\int_{\partial D} \big(u_1(x)\frac{\partial u_2(x)}{\partial \nu} - u_2(x)\frac{\partial u_1(x)}{\partial \nu}\big)dx,
\end{aligned} \tag{3.8}$$

(where dx in the right-hand side of (3.8) denotes the standard measure on ∂D in $\mathbb{R}^d$). (To start these transformations, we use that $\psi(x,k)$, $\psi_\gamma(x,k)$ and $\psi^+(x,k)$ satisfy (1.5) for $E = k^2$ and replace $v\psi$ in (3.1), (3.2) by $(\Delta + E)\psi$, $v\psi_\gamma$ in (3.3), (3.4) by $(\Delta + E)\psi_\gamma$ and $v\psi^+$ in (3.5), (3.6) by $(\Delta + E)\psi^+$.) However, these calculations can be shortened by means of the following Alessandrini identity (being valid under the assumptions of Theorem 1):

$$\int_D (v(x) - v^0(x))\psi(x)\psi^0(x)dx = \int_{\partial D}\int_{\partial D} \psi^0(x)(\Phi - \Phi^0)(x,y,E)\psi(y)dydx \tag{3.9}$$

for any ψ and ψ^0 such that ψ satisfies (1.1), ψ^0 satisfies (1.1) with v replaced by v^0 and where ψ and ψ^0 are sufficiently regular in D, for example, $\psi, \psi^0 \in H^1(D)$. In a slightly

different form the identity (3.9) was given in Lemma 1 of [A]. The proof of (3.9) is based on (3.8).

Formula (2.8) follows from (3.1), (3.9) and the fact that $\psi(x,k)$ satisfies (1.5) for $E=k^2$ and $\psi^0(x,-l)$ satisfies (1.5) with v replaced by v^0, for $E=l^2$. Formulas (2.9) with $x\in\mathbb{R}^d\backslash\bar D$ follow from (3.2) with $x\in\mathbb{R}^d\backslash\bar D$, (3.9) and the fact that $\psi(y,k)$ satisfies (1.5) in y for $E=k^2$ and that $R^0(x,y,k)$ for $x\in\mathbb{R}^d\backslash\bar D$ satisfies (1.5) in y in an open neighborhood of $\bar D$, with v replaced by v^0, for $E=k^2$. The latter statement about R^0 follows from (2.7) and the symmetry

$$R(x,y,k)=R(y,x,-k), \tag{3.10}$$

where $x,y\in\mathbb{R}^d$, $k\in\mathbb{C}^d\backslash\mathbb{R}^d$ (and R is defined by means of (2.2)-(2.4)). The symmetry (3.10) was found in [No4]. Finally, formulas (2.9) for $x\in\partial D$ arise as a limit of (2.9) with $x\in\mathbb{R}^d\backslash\bar D$.

The proof of (2.10)-(2.13) is similar to the proof of (2.8), (2.9).

Theorem 1 is proved.

Proof of Proposition 1. In this proof we obtain and use the following formula (being valid under the assumption (1.6) on v^0 and v):

$$h_\gamma(k,l)=h^0_\gamma(k,l)+\big(\frac{1}{2\pi}\big)^d\int_{\mathbb{R}^d}\psi^0_{-\gamma}(x,-k,-l)(v(x)-v^0(x))\psi_\gamma(x,k)dx \tag{3.11}$$

for $\gamma\in\mathbb{S}^{d-1}$, $k\in\mathbb{R}^d\backslash(0\cup\mathcal{E}^0_\gamma\cup\mathcal{E}_\gamma)$, $l\in\mathbb{R}^d$, $k^2=l^2$. Formula (2.15) follows from (3.11), (3.9) and the fact that $\psi_\gamma(x,k)$ satisfies (1.5) for $E=k^2$ and $\psi^0_{-\gamma}(x,-k,-l)$ satisfies (1.5) with v replaced by v^0, for $E=k^2=l^2$. Thus, to prove Proposition 1, it remains to prove (3.11).

To prove (3.11) we use, in particular, that

$$\psi_\gamma(x,k,l)=\psi^0_\gamma(x,k,l)+\int_{\mathbb{R}^d}R^0_\gamma(x,y,k)(v(y)-v^0(y))\psi_\gamma(y,k,l)dy \tag{3.12}$$

for $x\in\mathbb{R}^d$, $\gamma\in\mathbb{S}^{d-1}$, $k\in\mathbb{R}^d\backslash(0\cup\mathcal{E}^0_\gamma\cup\mathcal{E}_\gamma)$, $l\in\mathbb{R}^d$, $k^2=l^2$. To obtain (3.12) we write (2.14a) as

$$\begin{gathered}\psi_\gamma(x,k,l)-\int_{\mathbb{R}^d}G_\gamma(x-y,k)v^0(y)\psi_\gamma(y,k,l)dy-e^{ilx}=\\ \int_{\mathbb{R}^d}G_\gamma(x-y,k)(v(y)-v^0(y))\psi_\gamma(y,k,l)dy,\end{gathered} \tag{3.13}$$

where $x\in\mathbb{R}^d$, $\gamma\in\mathbb{S}^{d-1}$, $k\in\mathbb{R}^d\backslash(0\cup\mathcal{E}_\gamma)$, $l\in\mathbb{R}^d$, $k^2=l^2$. Replacing e^{ilx} in (3.13) by its expression from (2.14a) with v and ψ_γ replaced by v^0 and ψ^0_γ we have

$$\begin{gathered}\psi_\gamma(x,k,l)-\psi^0_\gamma(x,k,l)-\int_{\mathbb{R}^d}G_\gamma(x-y,k)v^0(y)(\psi_\gamma(y,k,l)-\psi^0_\gamma(y,k,l))dy=\\ \int_{\mathbb{R}^d}G_\gamma(x-y,k)(v(y)-v^0(y))\psi_\gamma(y,k,l)dy,\end{gathered} \tag{3.14}$$

where $x \in \mathbb{R}^d$, $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}^0_\gamma \cup \mathcal{E}_\gamma)$, $l \in \mathbb{R}^d$, $k^2 = l^2$.

Comparing (3.14) (as an equation for $\psi_\gamma - \psi^0_\gamma$) with the following equation (arising from (2.2), (2.5), (2.14b)) for $R^0_\gamma(x, y, k)$:

$$R^0_\gamma(x, y, k) - \int_{\mathbb{R}^d} G_\gamma(x - z, k) v^0(z) R^0_\gamma(z, y, k) dz = G_\gamma(x - y, k), \tag{3.15}$$

$x, y \in \mathbb{R}^d$, $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}^0_\gamma \cup \mathcal{E}_\gamma)$, we obtain (3.12).

Note that for $v \equiv 0$ equation (3.12) with k, γ and l replaced by $-k$, $-\gamma$ and $-l$ takes the form

$$e^{-ilx} = \psi^0_{-\gamma}(x, -k, -l) - \int_{\mathbb{R}^d} R^0_{-\gamma}(x, y, -k) v^0(y) e^{-ily} dy, \tag{3.16}$$

$x \in \mathbb{R}^d$, $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}^0_\gamma)$. To prove (3.11) we will use also the following symmetry (following from (3.10))

$$R_\gamma(x, y, k) = R_{-\gamma}(y, x, -k), \quad x, y \in \mathbb{R}^d, \quad \gamma \in \mathbb{S}^{d-1}, \quad k \in \mathbb{R}^d \backslash 0. \tag{3.17}$$

The following sequences of equalities proves (3.11):

$$\begin{aligned}
&(2\pi)^d h_\gamma(k, l) = \int_{\mathbb{R}^d} e^{-ilx} v(x) \psi_\gamma(x, k) dx = \int_{\mathbb{R}^d} e^{-ilx} v^0(x) \psi_\gamma(x, k) dx + \\
&\int_{\mathbb{R}^d} e^{-ilx} (v(x) - v^0(x)) \psi_\gamma(x, k) dx \overset{(3.16)}{=} \int_{\mathbb{R}^d} e^{-ilx} v^0(x) \psi_\gamma(x, k) dx + \\
&\int_{\mathbb{R}^d} \psi^0_{-\gamma}(x, -k, -l)(v(x) - v^0(x)) \psi_\gamma(x, k) dx - \\
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} R^0_{-\gamma}(x, y, -k) v^0(y) e^{-ily} (v(x) - v^0(x)) \psi_\gamma(x, k) dy dx \overset{(3.17)}{=} \\
&\int_{\mathbb{R}^d} e^{-ilx} v^0(x) \psi_\gamma(x, k) dx + \int_{\mathbb{R}^d} \psi^0_{-\gamma}(x, -k, -l)(v(x) - v^0(x)) \psi_\gamma(x, k) dx - \\
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} R^0_\gamma(y, x, k) v^0(y) e^{-ily} (v(x) - v^0(x)) \psi_\gamma(x, k) dx dy \overset{(3.4)}{=} \\
&\int_{\mathbb{R}^d} e^{-ilx} v^0(x) \psi_\gamma(x, k) dx + \int_{\mathbb{R}^d} \psi^0_{-\gamma}(x, -k, -l)(v(x) - v^0(x)) \psi_\gamma(x, k) dx + \\
&\int_{\mathbb{R}^d} e^{-ily} v^0(y) (\psi^0_\gamma(y, k) - \psi_\gamma(y, k)) dy = \\
&(2\pi)^d h^0_\gamma(k, l) + \int_{\mathbb{R}^d} \psi^0_{-\gamma}(x, -k, -l)(v(x) - v^0(x)) \psi_\gamma(x, k) dx
\end{aligned}$$

for $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \backslash (0 \cup \mathcal{E}^0_\gamma \cup \mathcal{E}_\gamma)$, $l \in \mathbb{R}^d$, $k^2 = l^2$.

Proposition 1 is proved.

Proof of Proposition 2. Let us prove Proposition 2 for the case of equation (2.9a). Consider the operator $A(k)$ of (2.9a):

$$A(k) = R^0(k) \left(\Phi(E) - \Phi^0(E)\right), \tag{3.18}$$

where $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, and where the operator $R^0(k)$ is defined by

$$R^0(k)\varphi(x) = \int_{\partial D} R^0(x, y, k)\varphi(y)dy, \quad x \in \partial D, \tag{3.19}$$

where φ is a test function.

Under the assumptions of Theorem 1, the operator $\Phi(E) - \Phi^0(E)$ is compact in $L^\infty(\partial D)$. This follows from the following properties of the Schwartz kernel of $\Phi(E)-\Phi^0(E)$:

$$(\Phi - \Phi^0)(x, y, E) \ \text{ is continuous for } \ x, y \in \partial D, \quad x \neq y, \tag{3.20}$$

$$\begin{aligned} &|(\Phi - \Phi^0)(x, y, E)| \leq C_1 |x - y|^{2-d}, \quad x, y \in \partial D, \quad \text{for} \quad d \geq 3, \\ &|(\Phi - \Phi^0)(x, y, E)| \leq C_1 |\ln|x - y||, \quad x, y \in \partial D, \quad \text{for} \quad d = 2, \end{aligned} \tag{3.21}$$

where C_1 is some constant (dependent on D, v, v^0, E and d). Note that for $v^0 \equiv 0$ the result that $\Phi(E) - \Phi^0(E)$ is compact in $L^\infty(\partial D)$ (under the assumptions of Theorem 1) was given in [No1].

If (1.2a) is fulfilled, v^0 satisfies (1.15) and $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, then $R^0(k)$ is a compact operator in $L^\infty(\partial D)$. This follows already from the following properties of $R^0(x, y, k)$:

$$R^0(x, y, k) \ \text{ is continuous for } \ x, y \in \bar{D}, \quad x \neq y, \tag{3.22}$$

$$\begin{aligned} &|R^0(x, y, k)| \leq C_2 |x - y|^{2-d}, \quad x, y \in \bar{D}, \quad \text{for} \quad d \geq 3, \\ &|R^0(x, y, k)| \leq C_2 |\ln|x - y||, \quad x, y \in \bar{D}, \quad \text{for} \quad d = 2, \end{aligned} \tag{3.23}$$

where C_2 is some constant (dependent on D, v^0, k and d).

Actually, under the aforementioned assumptions on D, v^0 and k, $R^0(k)$ is a bounded operator from $L^\infty(\partial D)$ to $C^\alpha(\partial D)$ for any $\alpha \in [0, 1[$ (where C^α denotes the Hölder space). This result in the general case follows from this result for $v^0 \equiv 0$ (when $R^0(x, y, k) = G(x - y, k)$), the relation (2.2), the estimate (3.23) and the property that

$$\begin{aligned} &\int_D G(x - z, k)u(z)dz \in C^\alpha(\mathbb{R}^d), \quad \text{at least, for any} \quad \alpha \in [0, 1] \\ &(\text{as a function of } x) \quad \text{for} \quad u \in L^\infty(D) \quad (\text{and} \quad k \in \mathbb{C}^d \backslash \mathbb{R}^d). \end{aligned}$$

The formula (3.18) and the aforementioned properties of the operators $\Phi(E) - \Phi^0(E)$ and $R^0(k)$ imply that, under the assumptions of Theorem 1, for fixed $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, $A(k)$ is a compact operator in $L^\infty(\partial D)$ and, thus, (2.9a) is a Fredholm linear integral equation of the second kind for ψ in $L^\infty(\partial D)$.

Under the assumptions of Theorem 1, for $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, the aforementioned properties of $\Phi(E) - \Phi^0(E)$ and $R^0(k)$ and the property that $\psi^0 \in C^\alpha(\partial D)$, $\alpha \in [0, 1[$, imply also that

$$\text{if} \quad \psi \in L^\infty(\partial D) \quad \text{satisfies} \quad (2.9a), \quad \text{then} \quad \psi \in C^\alpha(\partial D) \quad \text{for any} \quad \alpha \in [0, 1[. \tag{3.24}$$

Note also that if (1.2a) is fulfilled, v^0 satisfies (1.15) and $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, then $R^0(k)$ is a bounded operator from $L^2(\partial D)$ to $H^1(\partial D)$. One can prove this result in the general case proceeding from this result for $v^0 \equiv 0$ (when $R^0(x,y,k) = G(x-y,k)$), equation (2.2) with its iterations for $R(x,y,k)$, $R(x,y,k) - G(x-y,k)$ and so on, estimate (3.23) and the property that

$$\int_D G(x-z,k)u(z)dz \in H^2(D) \ \text{ (as a function of } \ x) \ \text{ for } \ u \in L^2(D) \ \text{ (and } \ k \in \mathbb{C}^d \backslash \mathbb{R}^d).$$

(For $v^0 \equiv 0$ this result was given, for example, in [Na1].) Therefore, under the assumptions of Theorem 1, for $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, one can see that

$$\text{if } \ \psi \in L^\infty(\partial D) \ \text{ satisfies } \ (2.9a), \ \text{ then also } \ \psi \in H^1(\partial D). \tag{3.25}$$

To prove Proposition 2 for the case of equation (2.9a) it remains to show that, under the assumptions of Theorem 1, equation (2.9a) at fixed $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, is uniquely solvable for $\psi \in L^\infty(\partial D)$ iff $k \notin \mathcal{E}$.

For $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, under the assumption (1.6) for v and v^0, due to (2.1a) and (3.7), $k \notin \mathcal{E}$ iff equation (3.2) is uniquely solvable for $\psi = e^{ikx}\mu(x,k)$ with $\mu \in L^\infty(\mathbb{R}^d)$. In turn, for $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, under the assumption (1.15) for v and v^0, equation (3.2) is uniquely solvable for $\psi = e^{ikx}\mu(x,k)$ with $\mu \in L^\infty(\mathbb{R}^d)$ iff (3.2) is uniquely solvable for $\psi \in C(\mathbb{R}^d)$. Thus, it remains to show that, under the assumptions of Theorem 1, equation (2.9a) at fixed $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, is uniquely solvable for $\psi \in L^\infty(\partial D)$ iff (3.2) is uniquely solvable for $\psi \in C(\mathbb{R}^d)$. This proof consists of the following two parts.

Part 1. Suppose that (under the assumptions of Theorem 1) at fixed $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, equation (3.2) has several solutions in $C(\mathbb{R}^d)$. Then, repeating the proof of (2.9) separately for each solution, we find that the restriction to ∂D of each of these solutions satisfies (2.9a). In addtion, different solutions ψ have different restrictions to ∂D. This follows from (1.3). Thus at fixed $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, equation (2.9a) has, at least, as many solutions as equation (3.2).

Part 2. To prove the converse, we use the following identities:

$$\begin{aligned}
&\int_D R^0(x,y,k)(v(y) - v^0(y))\psi(y)dy \overset{(1.1)}{=} \int_D R^0(x,y,k)(\Delta + E)\psi(y)dy - \\
&\int_D R^0(x,y,k)v^0(y)\psi(y)dy \overset{(3.8)}{=} \int_D \psi(y)(\Delta_y + E - v^0(y))R^0(x,y,k)dy + \\
&\int_{\partial D} \big(R^0(x,y,k)\frac{\partial}{\partial \nu_y}\psi(y) - \psi(y)\frac{\partial}{\partial \nu_y}R^0(x,y,k)\big)dy \overset{(2.7),(3.10)}{=} \\
&\int_D \psi(y)\delta(x-y)dy + \int_{\partial D} \big(R^0(x,y,k)\frac{\partial}{\partial \nu_y}\psi(y) - \psi(y)\frac{\partial}{\partial \nu_y}R^0(x,y,k)\big)dy \\
&\text{for } \ x \in \mathbb{R}^d \backslash \partial D,
\end{aligned} \tag{3.26}$$

$$\int_D R^0(x,y,k)(v(y) - v^0(y))\psi(y)dy \overset{(3.9),(2.9b)}{=} \int_{\partial D} A(x,y,k)\psi(y)dy \ \text{ for } \ x \in \mathbb{R}^d \backslash \bar{D}, \tag{3.27}$$

where $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, ψ satisfies (1.1) and is sufficiently regular in D, for example, $\psi \in H^1(D)$.

Let $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, be fixed. Suppose that $\psi \in L^\infty(\partial D)$ solves (2.9a). Due to (3.24), (3.25), we have also that $\psi \in C^\alpha(\partial D)$, $\alpha \in [0, 1[$, and $\psi \in H^1(\partial D)$. Consider this ψ as Dirichlet data for equation (1.1) and consider the solution ψ (of (1.1)) corresponding to these data. We have that $\psi \in C^\alpha(\bar{D})$, $\alpha \in [0, 1[$, and $\psi \in H^{3/2}(D)$.

Let ψ be also defined on $\mathbb{R}^d \backslash \bar{D}$ by (2.9) with $x \in \mathbb{R}^d \backslash \bar{D}$ (in terms of $\psi\big|_{\partial D}$). Let us prove that ψ defined in such a way on $\mathbb{R}^d = \partial D \cup D \cup (\mathbb{R}^d \backslash \bar{D})$ satisfies (3.2) and belongs to $C(\mathbb{R}^d)$. As a particular case of the aforementioned property $\psi \in C^\alpha(\bar{D})$, $\alpha \in [0, 1[$, we have that $\psi \in C(\bar{D})$. Proceeding from the definition of ψ on $\mathbb{R}^d \backslash \bar{D}$, one can easily show that, at least, $\psi \in C(\mathbb{R}^d \backslash D)$. The properties $\psi \in C(\bar{D})$ and $\psi \in C(\mathbb{R}^d \backslash D)$ imply that $\psi \in C(\mathbb{R}^d)$. The proof that ψ satisfies (3.2) consists in the following. First, from (2.9) with $x \in \mathbb{R}^d \backslash \bar{D}$ and (3.27) (and the continuity of ψ) we obtain that ψ satisfies (3.2) for $x \in \mathbb{R}^d \backslash D$. In addition, taking into account (3.26) we have also that

$$\psi(x) = \psi^0(x, k) + \int_{\partial D} \big(R^0(x, y, k) \frac{\partial}{\partial \nu_y} \psi(y) - \psi(y) \frac{\partial}{\partial \nu_y} R^0(x, y, k)\big) dy \tag{3.28}$$

for $x \in \mathbb{R}^d \backslash \bar{D}$, where $\frac{\partial}{\partial \nu}\psi$ is taken for ψ defined on $\bar{D}$. Further, (as well as for $v^0 \equiv 0$, see [No1],[Na2]) proceeding from (3.28) we obtain that

$$\psi(x) = \psi^0(x, k) + \psi(x) + \int_{\partial D} \big(R^0(x, y, k) \frac{\partial}{\partial \nu_y} \psi(y) - \psi(y) \frac{\partial}{\partial \nu_y} R^0(x, y, k)\big) dy \tag{3.29}$$

for $x \in D$, where $\frac{\partial}{\partial \nu}\psi$ is taken for ψ defined in $\bar{D}$. Note that proceeding from (3.28) and using that $\psi \in H^1(\partial D)$, $\frac{\partial}{\partial \nu}\psi \in L^2(\partial D)$ and the jump property of the double lager potential $\frac{\partial}{\partial \nu_y} R^0(x, y, k)$, we obtain, first, (3.29) in the limit for $x = \xi - 0\nu_\xi$, $\xi \in \partial D$ (where ν_ξ is the outward normal to ∂D at ξ). Further, using that $\psi^0(x, k)$ and $R^0(x, y, k)$ with $\frac{\partial}{\partial \nu_y} R^0(x, y, k)$, $y \in \partial D$, satisfy (1.1) with v^0 in place of v, for $E = k^2$, we obtain (3.29) for $x \in D$. Finally, from (3.29) and (3.26) we obtain that ψ satisfies (3.2) also for $x \in D$. Thus, any solution ψ of (2.9a) can be continued to a continuous solution of (3.2). This completes the (part 2 of) proof that, under the assumptions of Theorem 1, at fixed $k \in \mathbb{C}^d \backslash (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, equation (2.9a) is uniquely solvable for $\psi \in L^\infty(\partial D)$ iff (3.2) is uniquely solvable for $\psi \in C(\mathbb{R}^d)$.

The proof of Proposition 2 for the case of equation (2.9a) is completed. The proof of Proposition 2 for the cases of equations (2.11a) and (2.13a) is similar.

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